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Convolution and Differential Subordination for Multivalent Functions

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Abstract. The general classes of multivalent starlike, convex, close-to-convex and quasi-convex functions are introduced. These classes provide a unified treatment to various known subclasses. Inclusion and convolution properties are derived using the methods of convex hull and differential subordination.

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1. Motivation and preliminaries

Let $U = \{z : |z| < 1\}$ be the unit disk and $\mathcal{H}(U)$ be the class of all analytic functions defined on U. Let $\mathcal{A}_p$ be the class of all analytic functions of the form

$$f(z) = z^p + a_{p+1}z^{p+1} + \dots$$

with $\mathcal{A} := \mathcal{A}_1$. For two functions

$$f(z) = z^p + a_{p+1}z^{p+1} + \dots \quad \text{and} \quad g(z) = z^p + b_{p+1}z^{p+1} + \dots$$

in $\mathcal{A}_p$, the *Hadamard product* (or *convolution*) of f and g is the function $f * g$ defined by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

A function f is *subordinate* to F in U, written $f(z) \prec F(z)$, if there exists a Schwarz function w, analytic in U with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = F(w(z))$. If the function F is univalent in U, then $f(z) \prec F(z)$ is equivalent to $f(0) = F(0)$ and $f(U) \subseteq F(U)$.

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Let $\mathcal{S}^*$ and $\mathcal{K}$ respectively denote the subclasses of $\mathcal{A}$ consisting of starlike and convex functions in U. Recall that $f \in \mathcal{A}$ is convex if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in U),$$

and starlike if and only if

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0 \quad (z \in U).$$

These two classes and several other classes such as the classes of uniformly convex functions, starlike functions of order α, and strongly starlike functions investigated in geometric function theory are characterized by either of the quantities $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ lying in a given region in the right half-plane.

By the well-known Alexander theorem, $f \in \mathcal{K}$ if and only if $zf'(z) \in \mathcal{S}^*$. Since $zf'(z) = f(z) * (z/(1-z)^2)$, it follows that f is convex if and only if $f * g$ is starlike for $g(z) = z/(1-z)^2$. Moreover, since $f(z) = f(z) * (z/(1-z))$, the investigation of the classes of convex and starlike functions can be unified by considering the class of functions f for which $f * g$ is starlike for a fixed function g. These ideas motivated the investigation of the class of functions f for which

$$\frac{z(f*g)'(z)}{(f*g)(z)} \prec h(z),$$

where g is a fixed function in $\mathcal{A}$, and h is a convex function with positive real part. Shanmugam [38] introduced this class and several other related classes, and investigated inclusion and convolution properties by using the convex hull method [9, 36, 37] and the method of differential subordination.

Motivated by the investigation of Shanmugam [38], Ravichandran [29] and Ali *et al.* [1] (see also [3, 15, 23–25]), the following classes of multivalent functions will be studied. In the sequel, the function $g \in \mathcal{A}_p$ is assumed to be a fixed function, and unless otherwise stated, the function h is assumed to be a fixed normalized convex univalent function with positive real part and $h(0) = 1$.

Definition 1.1. *The class $\mathcal{S}_{p,g}(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying the condition $(g * f)(z)/z^p \neq 0$ in U and the subordination*

$$\frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)} \prec h(z).$$

Similarly, $\mathcal{K}_{p,g}(h)$ is the class of functions $f \in \mathcal{A}_p$ satisfying $(g * f)'(z)/z^{p-1} \neq 0$ in U and

$$\frac{1}{p}\left[1 + \frac{z(g*f)''(z)}{(g*f)'(z)}\right] \prec h(z).$$

With $g(z) = z^p/(1-z)$, the classes $\mathcal{S}_{p,g}(h) =: \mathcal{S}_p^*(h)$ and $\mathcal{K}_{p,g}(h) =: \mathcal{K}_p(h)$ consist respectively of all p-valent starlike and convex functions satisfying the respective subordinations

$$\frac{1}{p}\frac{zf'(z)}{f(z)} \prec h(z), \text{ and } \quad \frac{1}{p}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec h(z).$$

For these two classes, several interesting properties including distortion, growth and rotation inequalities as well as convolution properties have been investigated by Ali *et al.* [1]. Note that the two classes $\mathcal{S}_p^*(h)$ and $\mathcal{S}_{p,g}(h)$ are closely related; in fact, $f \in \mathcal{S}_{p,g}(h)$ if and only if $f * g \in \mathcal{S}_p^*(h)$. Similarly, $f \in \mathcal{K}_{p,g}(h)$ if and only if $f * g \in \mathcal{K}_p(h)$.

Definition 1.2. *The class $\mathcal{C}_{p,g}(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying the subordination*

$$\frac{1}{p}\frac{z(g*f)'(z)}{(g*\psi)(z)} \prec h(z)$$

for some $\psi \in \mathcal{S}_{p,g}(h)$.

Definition 1.3. *For any real number α, the class $\mathcal{K}_{p,g}^{\alpha}(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying $(g*f)(z)/z^p \neq 0$ and $(g*f)'(z)/z^{p-1} \neq 0$ in U, and the subordination*

$$\frac{\alpha}{p}\left[1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right]+\frac{(1-\alpha)}{p}\left[\frac{z(g*f)'(z)}{(g*f)(z)}\right] \prec h(z).$$

Definition 1.4. *The class $\mathcal{Q}_{p,g}(h)$ consists of functions $f \in \mathcal{A}_p$ satisfying the subordination*

$$\frac{1}{p}\frac{[z(g*f)'(z)]'}{(g*\phi)'(z)} \prec h(z)$$

for some $\phi \in \mathcal{K}_{p,g}(h)$.

Polya-Schoenberg [26] conjectured that the class of convex functions $\mathcal{K}$ is preserved under convolution with convex functions:

$$f, g \in \mathcal{K} \Rightarrow f * g \in \mathcal{K}.$$

In 1973, Ruscheweyh and Sheil-Small [36] proved the Polya-Schoenberg conjecture. In fact, they proved that the classes of convex functions, starlike functions and close-to-convex functions are closed under convolution with convex functions. For an interesting development on these ideas, see Ruscheweyh [37] (and also Duren [10, pp. 246–258], as well as Goodman [12, pp. 129–130]). Using the techniques developed in Ruscheweyh [37], several authors [1, 5, 7–9, 13, 14, 20–25, 27, 29, 34, 38–40] have proved that their classes are closed under convolution with convex (and other related) functions.

In the present paper, convolution properties as well as inclusion and related properties are investigated for the general classes of p-valent functions defined above. These classes are extension of the classes of convex, starlike, close-to-convex, α-convex, and quasi-convex functions. The results obtained here advanced known convolution properties of p-valent functions. For growth, distortion and related properties, see [1]. Corresponding results for meromorphic functions can be found in [2, 19].

The following definition and results are needed to prove our main results. For $\alpha \leq 1$, the class R_α of *prestarlike functions of order* α is defined by

$$R_\alpha := \left\{ f \in \mathcal{A} \ : \ f * \frac{z}{(1-z)^{2-2\alpha}} \in S^*(\alpha) \right\}$$

for $\alpha < 1$, and

$$R_1 := \left\{ f \in \mathcal{A} \ : \ \operatorname{Re}\frac{f(z)}{z} > \frac{1}{2} \right\}.$$

Theorem 1.1. [37, Theorem 2.4] *Let $\alpha \le 1$. If $f \in \mathcal{R}_\alpha$ and $g \in S^*(\alpha)$, then*

$$\frac{f * Hg}{f * g}(U) \subset \overline{co}(H(U))$$

for any analytic function $H \in \mathcal{H}(U)$, where $\overline{co}(H(U))$ denote the closed convex hull of $H(U)$.

Theorem 1.2. [18, Theorem 3.2a] *Let $\beta, \nu \in \mathbb{C}$, $h \in \mathcal{H}(U)$ be convex univalent in U, and $\operatorname{Re}(\beta h(z) + \nu) > 0$. If p is analytic in U with $p(0) = h(0)$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec h(z) \quad \Longrightarrow \quad p(z) \prec h(z).$$

Theorem 1.3. [18, Theorem 3.2b] *Let $h \in \mathcal{H}(U)$ be convex univalent in U with $h(0) = a$. Suppose that the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \nu} = h(z)$$

has a univalent solution q that satisfies $q(z) \prec h(z)$. If $p(z) = a + a_1 z + \cdots$ satisfies

$$p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec h(z),$$

then $p(z) \prec q(z)$, and q is the best dominant.

Theorem 1.4. [18, Theorem 3.1a] *Let h be convex in U, and $P : U \to \mathbb{C}$ with $\operatorname{Re} P(z) > 0$. If p is analytic in U, then*

$$p(z) + P(z)zp'(z) \prec h(z) \quad \Longrightarrow \quad p(z) \prec h(z).$$

2. Inclusion and convolution theorems

Every convex univalent function is starlike or equivalently $\mathcal{K} \subset \mathcal{S}^*$, and Alexander's theorem gives $f \in \mathcal{K}$ if and only if $zf' \in \mathcal{S}^*$. These properties remain valid even for multivalent functions.

Theorem 2.1. *Let g be a fixed function in $\mathcal{A}_p$ and h be a convex univalent function with positive real part and $h(0) = 1$. Then*

(i) $\mathcal{K}_{p,g}(h) \subseteq \mathcal{S}_{p,g}(h)$,
(ii) *$f \in \mathcal{K}_{p,g}(h)$ if and only if $\frac{1}{p} zf' \in \mathcal{S}_{p,g}(h)$.*

Proof. (i) Since h is a function with positive real part, it is clear that the function $f * g$ is p-valent convex and hence it is also p-valent starlike. Since $(f * g)(z)/z^p \neq 0$, the function q defined by

$$q(z) := \frac{1}{p} \frac{z(g * f)'(z)}{(g * f)(z)}$$

is analytic in U and satisfies

$$q(z) + \frac{1}{p} \frac{zq'(z)}{q(z)} = \frac{1}{p} \left(1 + \frac{z(g * f)''(z)}{(g * f)'(z)} \right). \tag{2.1}$$

If $f \in \mathcal{K}_{p,g}(h)$, the right-hand side of (2.1) is subordinate to h. It follows from Theorem 1.2 that $q(z) \prec h(z)$, and thus $\mathcal{K}_{p,g}(h) \subseteq \mathcal{S}_{p,g}(h)$.

(ii) Since

$$\frac{1}{p}\left(1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right)=\frac{1}{p}\frac{[z(g*f)'(z)]'(z)}{(g*f)'(z)}$$
$$=\frac{1}{p}\frac{z(g*\frac{1}{p}zf')'(z)}{(g*\frac{1}{p}zf')(z)},$$

it follows that $f\in\mathcal{K}_{p,g}(h)$ if and only if $\frac{1}{p}zf'\in\mathcal{S}_{p,g}(h)$. ∎

Suppose that h is convex univalent in U with $h(0)=1$ and that the differential equation

$$q(z)+\frac{1}{p}\frac{zq'(z)}{q(z)}=h(z)$$

has a univalent solution q that satisfies $q(z)\prec h(z)$. If $f\in\mathcal{K}_{p,g}(h)$, then from Theorem 1.3 and (2.1), it follows that $f\in\mathcal{S}_{p,g}(q)$, or equivalently $\mathcal{K}_{p,g}(h)\subset\mathcal{S}_{p,g}(q)$.

Theorem 2.2. *Let h be a convex univalent function satisfying the condition*

$$\operatorname{Re}h(z)>1-\frac{1-\alpha}{p}\quad(0\le\alpha<1),\tag{2.2}$$

and $\phi\in\mathcal{A}_p$ with $\phi/z^{p-1}\in\mathcal{R}_\alpha$. If $f\in\mathcal{S}_{p,g}(h)$, then $\phi*f\in\mathcal{S}_{p,g}(h)$.

Proof. For a function $f\in\mathcal{S}_{p,g}(h)$, let the function H be defined by

$$H(z):=\frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)}.$$

Then the function H is analytic in U and $H(z)\prec h(z)$. The function Φ defined by $\Phi(z):=\phi(z)/z^{p-1}$ belongs to $\mathcal{R}_\alpha$. We now show that $G(z):=(f*g)(z)/z^{p-1}$ is in $\mathcal{S}^*(\alpha)$. Since $f\in\mathcal{S}_{p,g}(h)$, and h is a convex univalent function satisfying (2.2), it follows that

$$\frac{1}{p}\operatorname{Re}\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right)>1-\frac{1-\alpha}{p},$$

and hence

$$\operatorname{Re}\frac{zG'(z)}{G(z)}=\operatorname{Re}\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right)-p+1>\alpha.$$

Thus $G\in\mathcal{S}^*(\alpha)$. Since $\Phi\in\mathcal{R}_\alpha$, $G\in\mathcal{S}^*(\alpha)$, and h is convex, an application of Theorem 1.1 shows that

$$\frac{(\Phi*GH)(z)}{(\Phi*G)(z)}\prec h(z).\tag{2.3}$$

The relations

$$z(g*f)'(z)=(g*zf')(z)\quad\text{and}\quad(g*f)(z)=z^{p-1}\left(\frac{g}{z^{p-1}}*\frac{f}{z^{p-1}}\right)(z)$$

yield

$$\begin{aligned}\frac{1}{p}\frac{z(g*\phi*f)'(z)}{(g*\phi*f)(z)} &= \frac{\phi(z)*\frac{1}{p}z(g*f)'(z)}{\phi(z)*(g*f)(z)}\\ &= \frac{\frac{\phi(z)}{z^{p-1}}*\frac{(g*f)(z)}{z^{p-1}}H(z)}{\frac{\phi(z)}{z^{p-1}}*\frac{(g*f)(z)}{z^{p-1}}}\\ &= \frac{(\Phi*GH)(z)}{(\Phi*G)(z)}.\end{aligned}$$

Thus the subordination (2.3) gives

$$\frac{1}{p}\frac{z(g*\phi*f)'(z)}{(g*\phi*f)(z)} \prec h(z),$$

which proves $\phi*f\in\mathcal{S}_{p,g}(h)$. ∎

Corollary 2.1. *Let h and ϕ satisfy the conditions of Theorem 2.2. Then $\mathcal{S}_{p,g}(h)\subseteq\mathcal{S}_{p,\phi*g}(h)$.*

Proof. If $f\in\mathcal{S}_{p,g}(h)$, Theorem 2.2 yields $f*\phi\in\mathcal{S}_{p,g}(h)$, that is $f*\phi*g\in\mathcal{S}_p^*(h)$. Hence $f\in\mathcal{S}_{p,\phi*g}(h)$. ∎

In particular, when $g(z)=z^p/(1-z)$, the following corollary is obtained:

Corollary 2.2. *Let h and ϕ satisfy the conditions of Theorem 2.2. If $f\in\mathcal{S}_p^*(h)$, then $f\in\mathcal{S}_{p,\phi}^*(h)$.*

Corollary 2.3. *Let h and ϕ satisfy the conditions of Theorem 2.2. If $f\in\mathcal{K}_{p,g}(h)$, then $f*\phi\in\mathcal{K}_{p,g}(h)$ and $\mathcal{K}_{p,g}(h)\subseteq\mathcal{K}_{p,\phi*g}(h)$.*

Proof. If $f\in\mathcal{K}_{p,g}(h)$, it follows from Theorem 2.1(ii) and Theorem 2.2, that $(zf'*\phi)/p\in\mathcal{S}_{p,g}(h)$. Hence $f*\phi\in\mathcal{K}_{p,g}(h)$. The second part follows from Corollary 2.1. ∎

Theorem 2.3. *Let h and ϕ satisfy the conditions of Theorem 2.2. If $f\in\mathcal{C}_{p,g}(h)$ with respect to $f_1\in\mathcal{S}_{p,g}(h)$, then $\phi*f\in\mathcal{C}_{p,g}(h)$ with respect to $\phi*f_1\in\mathcal{S}_{p,g}(h)$.*

Proof. As in the proof of Theorem 2.2, define the functions H, Φ and G by

$$H(z):=\frac{1}{p}\frac{z(g*f)'(z)}{(g*f_1)(z)},\quad \Phi(z):=\frac{\phi(z)}{z^{p-1}},\quad\text{and}\quad G(z):=\frac{(f_1*g)(z)}{z^{p-1}}.$$

Then $\Phi\in\mathcal{R}_\alpha$ and $G\in\mathcal{S}^*(\alpha)$. An application of Theorem 1.1 shows that the quantity $(\Phi*GH)(z)/(\Phi*G)(z)$ lies in the closed convex hull of $H(U)$. Since $h(z)$ is convex and $H\prec h$, it follows that

$$\frac{(\Phi*GH)(z)}{(\Phi*G)(z)}\prec h(z). \tag{2.4}$$

A direct calculation shows that

$$\begin{aligned}\frac{1}{p}\frac{z(g*\phi*f)'(z)}{(g*\phi*f_1)(z)} &= \frac{\phi(z)*\frac{1}{p}z(g*f)'(z)}{\phi(z)*(g*f_1)(z)}\\ &= \frac{\frac{\phi(z)}{z^{p-1}}*\frac{(g*f_1)(z)}{z^{p-1}}H(z)}{\frac{\phi(z)}{z^{p-1}}*\frac{(g*f_1)(z)}{z^{p-1}}}\end{aligned}$$

$$= \frac{(\Phi * GH)(z)}{(\Phi * G)(z)}.$$

Thus, the subordination (2.4) shows that $\phi * f \in \mathcal{C}_{p,g}(h)$ with respect to $\phi * f_1 \in \mathcal{S}_{p,g}(h)$. ▮

Corollary 2.4. *If h and ϕ satisfy the conditions of Theorem 2.2, then $\mathcal{C}_{p,g}(h) \subseteq \mathcal{C}_{p,\phi*g}(h)$.*

Proof. From Theorem 2.3, for a function $f \in \mathcal{C}_{p,g}(h)$ with respect to $f_1 \in \mathcal{S}_{p,g}(h)$, clearly

$$\frac{1}{p}\frac{z(g*\phi*f)'(z)}{(g*\phi*f_1)(z)} \prec h(z).$$

Thus $f \in \mathcal{C}_{p,\phi*g}(h)$, and hence $\mathcal{C}_{p,g}(h) \subseteq \mathcal{C}_{p,\phi*g}(h)$. ▮

Theorem 2.4. *Let h be a convex univalent function with positive real part and $h(0) = 1$. Then*

(i) $\mathcal{K}^{\alpha}_{p,g}(h) \subseteq \mathcal{S}_{p,g}(h)$ *for* $\alpha > 0$,

(ii) $\mathcal{K}^{\alpha}_{p,g}(h) \subseteq \mathcal{K}^{\beta}_{p,g}(h)$ *for* $\alpha > \beta \geq 0$.

Proof. (i) Let

$$J_{p,g}(\alpha; f(z)) := \frac{\alpha}{p}\left[1 + \frac{z(g*f)''(z)}{(g*f)'(z)}\right] + \frac{(1-\alpha)}{p}\left[\frac{z(g*f)'(z)}{(g*f)(z)}\right]$$

and the function $q(z)$ be defined by

$$q(z) := \frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)}.$$

A computation yields

$$J_{p,g}(\alpha; f(z)) = q(z) + \frac{\alpha z q'(z)}{pq(z)}.$$

Let $f \in \mathcal{K}^{\alpha}_{p,g}(h)$, so that $J_{p,g}(\alpha; f(z)) \prec h(z)$. Now an application of Theorem 1.2 shows that $q(z) \prec h(z)$. Hence $f \in \mathcal{S}_{p,g}(h)$.

(ii) The case $\beta = 0$ is contained in (i), and so we assume $\beta > 0$. Now,

$$\begin{aligned} J_{p,g}(\beta; f(z)) &= \frac{(1-\beta)}{p}\frac{z(g*f)'(z)}{(g*f)(z)} + \frac{\beta}{p}\left(1 + \frac{z(g*f)''(z)}{(g*f)'(z)}\right) \\ &= (1 - \frac{\beta}{\alpha})\frac{z(g*f)'(z)}{p(g*f)(z)} + \frac{\beta}{\alpha}J_{p,g}(\alpha; f(z)). \end{aligned}$$

From part (i),

$$\frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)} \prec h(z)$$

and

$$J_{p,g}(\alpha; f(z)) \prec h(z).$$

Hence $J_{p,g}(\beta; f(z)) \prec h(z)$, proving that $f \in \mathcal{K}^{\beta}_{p,g}(h)$. ▮

Theorem 2.5. *Let h be a convex univalent function with positive real part and $h(0) = 1$. Then*

(i) $\mathcal{K}_{p,g}(h) \subseteq \mathcal{Q}_{p,g}(h) \subseteq \mathcal{C}_{p,g}(h)$,

(ii) $f \in \mathcal{Q}_{p,g}(h)$ *if and only if* $\frac{1}{p}zf' \in \mathcal{C}_{p,g}(h)$.

Proof. (i) By taking $f = \phi$, it follows from the definition that $\mathcal{K}_{p,g}(h) \subseteq \mathcal{Q}_{p,g}(h)$. To prove the second inclusion, let

$$q(z) = \frac{1}{p}\frac{z(g*f)'(z)}{(g*\phi)(z)}.$$

Computations show that

$$q(z) + \frac{zq'(z)}{\frac{z(g*\phi)'(z)}{(g*\phi)(z)}} = \frac{1}{p}\frac{[z(g*f)'(z)]'(z)}{(g*\phi)'(z)}. \tag{2.5}$$

If $f \in \mathcal{Q}_{p,g}(h)$, then there exists a function $\phi \in \mathcal{K}_{p,g}(h)$ such that the expression on the right-hand side of (2.5) is subordinate to h. Also $\phi \in \mathcal{K}_{p,g}(h) \subseteq \mathcal{S}_{p,g}(h)$ implies

$$\operatorname{Re}\frac{z(g*\phi)'(z)}{(g*\phi)(z)} > 0.$$

Hence, an application of Theorem 1.4 to (2.5) yields $q(z) \prec h(z)$. This shows that $f \in \mathcal{C}_{p,g}(h)$.

(ii) It is easy to see that

$$\frac{1}{p}\frac{[z(g*f)'(z)]'(z)}{(g*\phi)'(z)} = \frac{1}{p}\frac{z(g*\frac{1}{p}zf')'(z)}{(g*\frac{1}{p}z\phi')(z)}. \tag{2.6}$$

Now if $f \in \mathcal{Q}_{p,g}(h)$ with respect to a function $\phi \in \mathcal{K}_{p,g}(h)$, then the expression on the left-hand side of (2.6) is subordinate to h. Now by Theorem 2.1(ii) and hence by definition of $\mathcal{C}_{p,g}(h)$, $\frac{1}{p}zf' \in \mathcal{C}_{p,g}(h)$.

Conversely, if $\frac{1}{p}zf' \in \mathcal{C}_{p,g}(h)$, then there exists a function $\phi_1 \in \mathcal{S}_{p,g}(h)$ such that $\frac{1}{p}z\phi' = \phi_1$. The expression on the right-hand side of (2.6) is subordinate to h and thus $f \in \mathcal{Q}_{p,g}(h)$. ∎

Corollary 2.5. *Let h and ϕ satisfy the conditions of Theorem 2.2. If $f \in \mathcal{Q}_{p,g}(h)$, then $\phi * f \in \mathcal{Q}_{p,g}(h)$.*

Proof. If $f \in \mathcal{Q}_{p,g}(h)$, by Theorem 2.5(ii), $\frac{1}{p}zf' \in \mathcal{C}_{p,g}(h)$. Theorem 2.3 shows that $\frac{1}{p}z(\phi*f)' \in \mathcal{C}_{p,g}(h)$. From Theorem 2.5(ii), $\phi * f \in \mathcal{Q}_{p,g}(h)$. ∎

Corollary 2.6. *If h and ϕ satisfy the conditions of Theorem 2.2, then $\mathcal{Q}_{p,g}(h) \subseteq \mathcal{Q}_{p,\phi*g}$.*

Proof. If $f \in \mathcal{Q}_{p,g}(h)$, Corollary 2.5 yields $\phi * f \in \mathcal{Q}_{p,g}(h)$ with respect to $\phi * \psi \in \mathcal{K}_{p,g}(h)$. The subordination

$$\frac{1}{p}\frac{[z(g*\phi*f)'(z)]'}{(g*\phi*\psi)'(z)} \prec h(z)$$

gives $f \in \mathcal{Q}_{p,g*\phi}$. Therefore, $\mathcal{Q}_{p,g}(h) \subseteq \mathcal{Q}_{p,g*\phi}$. ∎

A function f is prestarlike of order 0 if $f(z)*(z/(1-z)^2)$ is starlike, or equivalently if f is convex. Thus, the class of prestarlike functions of order 0 is the class of convex functions, and therefore the results obtained in this paper contain those of Shanmugam [38] for the special case $p = 1$ and $\alpha = 0$.

Example 2.1. Let $p = 1$, $g(z) = z/(1-z)$, and $\alpha = 0$. For $h(z) = (1+z)/(1-z)$, Theorem 2.1 reduces to the following: $\mathcal{K} \subseteq \mathcal{S}^*$ and $f \in \mathcal{K} \Leftrightarrow zf' \in \mathcal{S}^*$. Also Theorem 2.2 reduces to $f \in \mathcal{S}^*, \phi \in \mathcal{K} \Rightarrow f * \phi \in \mathcal{S}^*$, and Corollary 2.3 shows that the class of convex functions is closed under convolution with convex functions.

For

$$h(z) = 1 + \frac{2}{\pi^2}\left[\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right]^2,$$

the results obtained imply that $\mathcal{UCV} \subseteq \mathcal{S}_p$ and $f \in \mathcal{UCV} \Leftrightarrow zf' \in \mathcal{S}_p$, where $\mathcal{UCV}$ and $\mathcal{S}_p$ are the classes of uniformly convex functions and parabolic starlike functions [34,35]. It also follows as special cases that the classes $\mathcal{S}_p$ and $\mathcal{UCV}$ are closed under convolution with convex functions. For other related results for uniformly convex functions, see [4,6,16,17,27,30–33].

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